Cubic Splines

Given a function f defined on $[a, b]$ and a set of nodes $a = x_0 < x_1 < \cdots < x_n = b$, a **cubic spline interpolant** S for f is a function that satisfies the following conditions:

(a) $S(x)$ is a cubic polynomial, denoted $S_j(x)$, on the subinterval $[x_j, x_{j+1}]$ for each $j = 0, 1, \ldots, n-1$;

(b) $S_j(x_j) = f(x_j)$ and $S_j(x_{j+1}) = f(x_{j+1})$ for each $j = 0, 1, \ldots, n-1$;

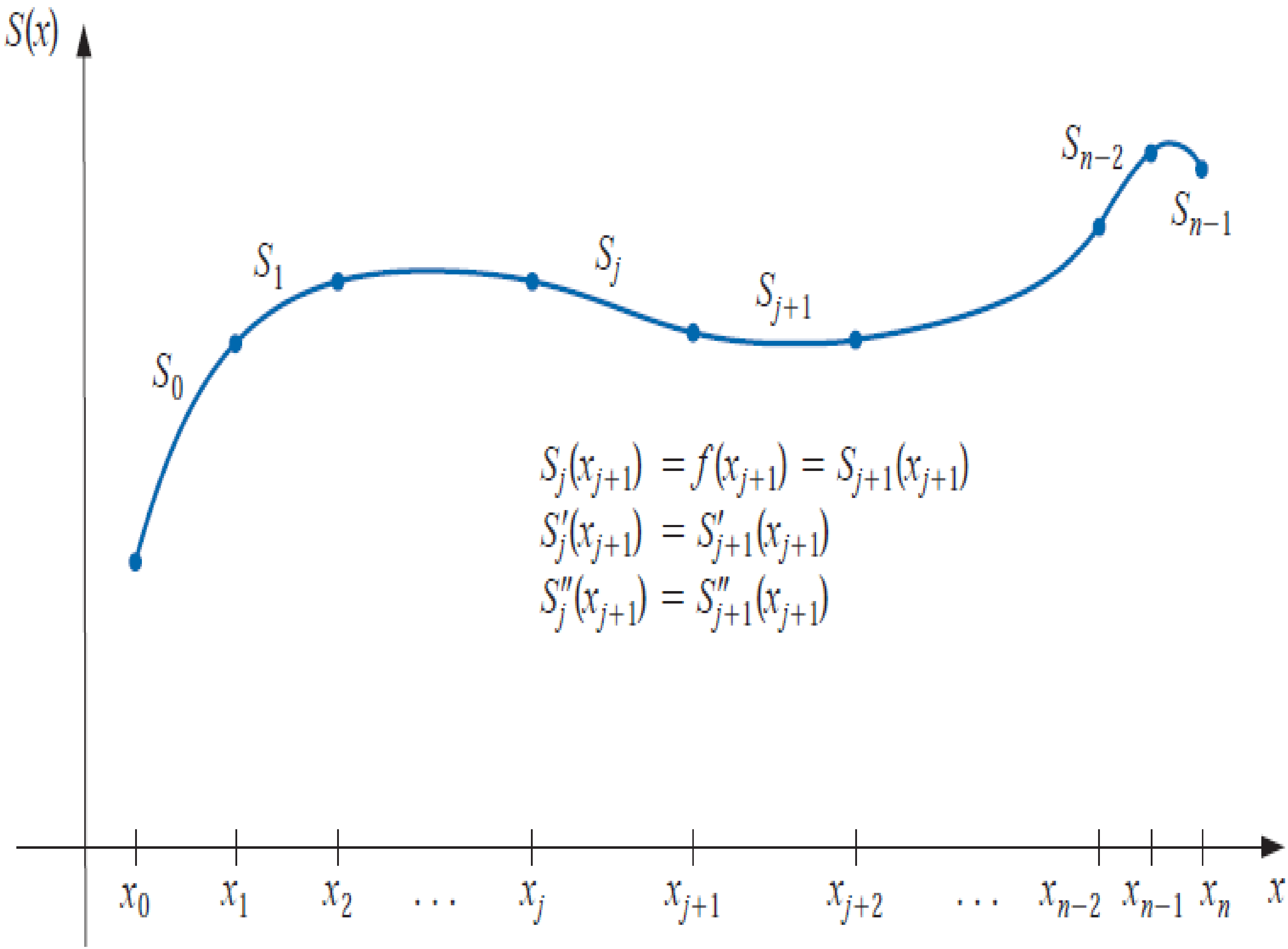
(c) $S_{j+1}(x_{j+1}) = S_j(x_{j+1})$ for each $j = 0, 1, \ldots, n-2$; (*Implied by* (**b**).)

(d) $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$ for each $j = 0, 1, \ldots, n-2$;

(e) $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$ for each $j = 0, 1, \ldots, n-2$;

(f) One of the following sets of boundary conditions is satisfied:

- **(i)** $S''(x_0) = S''(x_n) = 0$ (**natural** (*or free*) **boundary**);
- **(ii)** $S'(x_0) = f'(x_0)$ and $S'(x_n) = f'(x_n)$ (**clamped boundary**). ■



$S(x)$
S_0
S_1
S_j
S_{j+1}
S_{n-2}
S_{n-1}
$S_j(x_{j+1}) = f(x_{j+1}) = S_{j+1}(x_{j+1})$
$S'_j(x_{j+1}) = S'_{j+1}(x_{j+1})$
$S''_j(x_{j+1}) = S''_{j+1}(x_{j+1})$
x_0 x_1 x_2 ... x_j x_{j+1} x_{j+2} ... x_{n-2} x_{n-1} x_n x

Example

Construct a natural cubic spline that passes through the points $(1, 2)$, $(2, 3)$, and $(3, 5)$.

Solution This spline consists of two cubics. The first for the interval $[1, 2]$, denoted

$$S_0(x) = a_0 + b_0(x-1) + c_0(x-1)^2 + d_0(x-1)^3,$$

and the other for $[2, 3]$, denoted

$$S_1(x) = a_1 + b_1(x-2) + c_1(x-2)^2 + d_1(x-2)^3.$$

There are 8 constants to be determined, which requires 8 conditions. Four conditions come from the fact that the splines must agree with the data at the nodes. Hence

$$2 = f(1) = a_0, \quad 3 = f(2) = a_0 + b_0 + c_0 + d_0, \quad 3 = f(2) = a_1, \quad \text{and}$$

$$5 = f(3) = a_1 + b_1 + c_1 + d_1.$$

Two more come from the fact that $S_0'(2) = S_1'(2)$ and $S_0''(2) = S_1''(2)$. These are

$$S_0'(2) = S_1'(2): \quad b_0 + 2c_0 + 3d_0 = b_1 \qquad \text{and} \qquad S_0''(2) = S_1''(2): \quad 2c_0 + 6d_0 = 2c_1$$

The final two come from the natural boundary conditions:

$$S_0''(1) = 0: \quad 2c_0 = 0 \qquad \text{and} \qquad S_1''(3) = 0: \quad 2c_1 + 6d_1 = 0.$$

Solving this system of equations gives the spline

$$S(x) = \begin{cases} 2 + \frac{3}{4}(x-1) + \frac{1}{4}(x-1)^3, \text{ for } x \in [1,2] \\ 3 + \frac{3}{2}(x-2) + \frac{3}{4}(x-2)^2 - \frac{1}{4}(x-2)^3, \text{ for } x \in [2,3] \end{cases}$$

■

Construction of a Cubic Spline

$$S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3 \qquad j = 0, 1, \ldots, n-1$$

$$S_j(x_j) = a_j = f(x_j)$$

Applying condition (**c**) in definition of a cubic spline:

$$a_{j+1} = S_{j+1}(x_{j+1}) = S_j(x_{j+1}) = a_j + b_j(x_{j+1} - x_j) + c_j(x_{j+1} - x_j)^2 + d_j(x_{j+1} - x_j)^3$$

$j = 0, 1, \ldots, n-2$

$$h_j = x_{j+1} - x_j \qquad j = 0, 1, \ldots, n-1$$

we also define $a_n = f(x_n)$,

$$a_{j+1} = a_j + b_j h_j + c_j h_j^2 + d_j h_j^3 \qquad (3.15)$$

$j = 0, 1, \ldots, n-1$

define $b_n = S'(x_n)$

$$S_j'(x) = b_j + 2c_j(x - x_j) + 3d_j(x - x_j)^2$$

$$S_j'(x_j) = b_j \qquad j = 0, 1, \ldots, n-1$$

Applying condition **(d)** gives

$$b_{j+1} = b_j + 2c_jh_j + 3d_jh_j^2, \qquad (3.16)$$

$j = 0, 1, \ldots, n-1$

define $c_n = S''(x_n)/2$

applying condition **(e)**

$$c_{j+1} = c_j + 3d_jh_j. \qquad (3.17)$$

$j = 0, 1, \ldots, n-1$

Solving for d_j in Eq. (3.17) and substituting this value into Eqs. (3.15) and (3.16) gives, for each $j = 0, 1, \ldots, n-1$, the new equations

$$a_{j+1} = a_j + b_j h_j + \frac{h_j^2}{3}(2c_j + c_{j+1}) \tag{3.18}$$

and

$$b_{j+1} = b_j + h_j(c_j + c_{j+1}). \tag{3.19}$$

The final relationship involving the coefficients is obtained by solving the appropriate equation in the form of equation (3.18), first for b_j,

$$b_j = \frac{1}{h_j}(a_{j+1} - a_j) - \frac{h_j}{3}(2c_j + c_{j+1}), \tag{3.20}$$

and then, with a reduction of the index, for b_{j-1}. This gives

$$b_{j-1} = \frac{1}{h_{j-1}}(a_j - a_{j-1}) - \frac{h_{j-1}}{3}(2c_{j-1} + c_j).$$

Substituting these values into the equation derived from Eq. (3.19), with the index reduced by one, gives the linear system of equations

$$h_{j-1}c_{j-1} + 2(h_{j-1} + h_j)c_j + h_j c_{j+1} = \frac{3}{h_j}(a_{j+1} - a_j) - \frac{3}{h_{j-1}}(a_j - a_{j-1}), \qquad (3.21)$$

$j = 1, 2, \ldots, n-1$

This system involves only the $\{c_j\}_{j=0}^{n}$ as unknowns.

So once the values of $\{c_j\}_{j=0}^{n}$ are determined, it is a simple matter to find $\{b_j\}_{j=0}^{n-1}$ from Eq. (3.20) and $\{d_j\}_{j=0}^{n-1}$ from Eq. (3.17).

The major question that arises is whether the values of $\{c_j\}_{j=0}^{n}$ can be found using the system of equations given in (3.21)

Natural Splines

Theorem

If f is defined at $a = x_0 < x_1 < \cdots < x_n = b$, then f has a unique natural spline interpolant S on the nodes $x_0, x_1, \ldots, x_n$; that is, a spline interpolant that satisfies the natural boundary conditions $S''(a) = 0$ and $S''(b) = 0$. ■

Proof The boundary conditions in this case imply that $c_n = S''(x_n)/2 = 0$ and that

$$0 = S''(x_0) = 2c_0 + 6d_0(x_0 - x_0),$$

so $c_0 = 0$. The two equations $c_0 = 0$ and $c_n = 0$ together with the equations in (3.21) produce a linear system described by the vector equation $A\mathbf{x} = \mathbf{b}$, where A is the $(n+1) \times (n+1)$ matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ h_0 & 2(h_0 + h_1) & h_1 & \ddots & & \vdots \\ 0 & h_1 & 2(h_1 + h_2) & h_2 & \ddots & 0 \\ \vdots & & \ddots & h_{n-2} & 2(h_{n-2} + h_{n-1}) & h_{n-1} \\ 0 & \cdots & \cdots & 0 & 0 & 1 \end{bmatrix},$$

and **b** and **x** are the vectors

$$\mathbf{b} = \begin{bmatrix} 0 \\ \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ \vdots \\ \frac{3}{h_{n-1}}(a_n - a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}) \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

Matrix A is strictly diagonally dominant. A linear system with a matrix of this form has a unique Solution.

Example

Use the data points $(0, 1)$, $(1, e)$, $(2, e^2)$, and $(3, e^3)$ to form a natural spline $S(x)$ that approximates $f(x) = e^x$.

Solution

$$S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3 \qquad j = 0, 1, \ldots, n - 1$$

$$n = 3, h_0 = h_1 = h_2 = 1, a_0 = 1, a_1 = e, a_2 = e^2, \text{ and } a_3 = e^3$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 3(e^2 - 2e + 1) \\ 3(e^3 - 2e^2 + e) \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$A\mathbf{x} = \mathbf{b}$ is equivalent to the system of equations

$$c_0 = 0,$$

$$c_0 + 4c_1 + c_2 = 3(e^2 - 2e + 1),$$

$$c_1 + 4c_2 + c_3 = 3(e^3 - 2e^2 + e),$$

$$c_3 = 0.$$

This system has the solution $c_0 = c_3 = 0$, and

$$c_1 = \frac{1}{5}(-e^3 + 6e^2 - 9e + 4) \approx 0.75685$$

$$c_2 = \frac{1}{5}(4e^3 - 9e^2 + 6e - 1) \approx 5.83007$$

$$b_j = \frac{1}{h_j}(a_{j+1} - a_j) - \frac{h_j}{3}(2c_j + c_{j+1}), \tag{3.20}$$

$$b_0 = \frac{1}{h_0}(a_1 - a_0) - \frac{h_0}{3}(c_1 + 2c_0)$$

$$= (e - 1) - \frac{1}{15}(-e^3 + 6e^2 - 9e + 4) \approx 1.46600,$$

$$b_1 = \frac{1}{h_1}(a_2 - a_1) - \frac{h_1}{3}(c_2 + 2c_1)$$

$$= (e^2 - e) - \frac{1}{15}(2e^3 + 3e^2 - 12e + 7) \approx 2.22285,$$

$$b_2 = \frac{1}{h_2}(a_3 - a_2) - \frac{h_2}{3}(c_3 + 2c_2)$$

$$= (e^3 - e^2) - \frac{1}{15}(8e^3 - 18e^2 + 12e - 2) \approx 8.80977$$

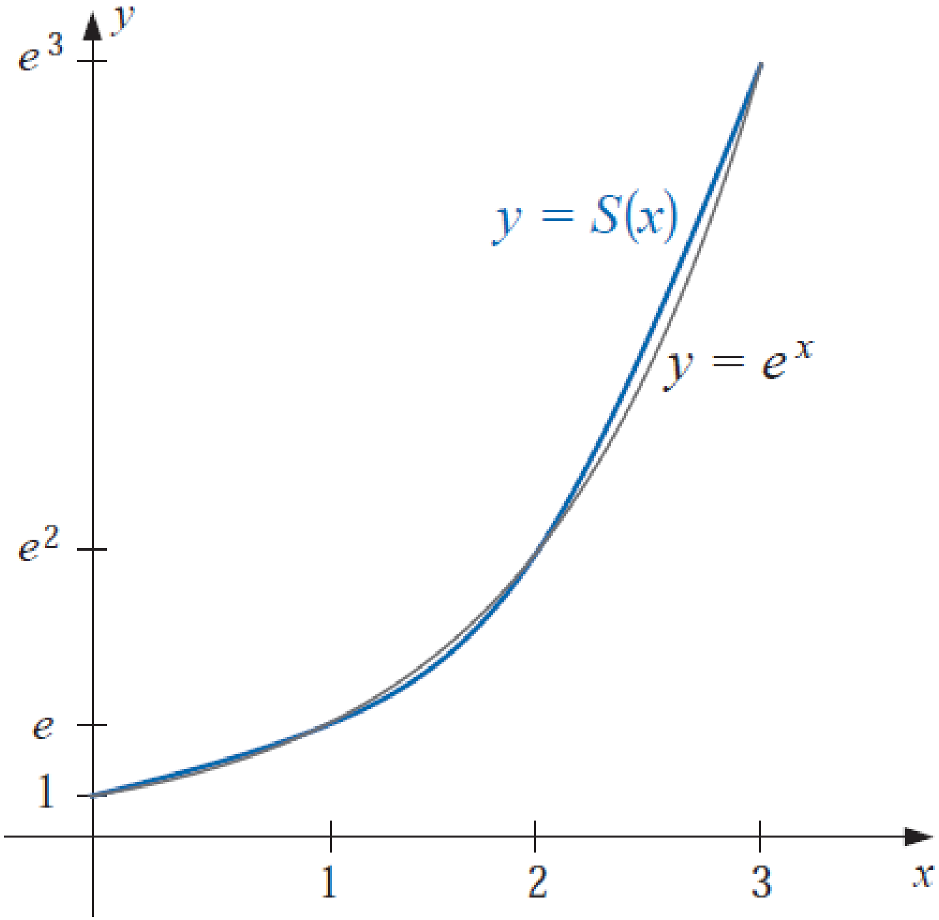
$$c_{j+1} = c_j + 3d_jh_j. \tag{3.17}$$

$$d_0 = \frac{1}{3h_0}(c_1 - c_0) = \frac{1}{15}(-e^3 + 6e^2 - 9e + 4) \approx 0.25228,$$

$$d_1 = \frac{1}{3h_1}(c_2 - c_1) = \frac{1}{3}(e^3 - 3e^2 + 3e - 1) \approx 1.69107,$$

$$d_2 = \frac{1}{3h_2}(c_3 - c_1) = \frac{1}{15}(-4e^3 + 9e^2 - 6e + 1) \approx -1.94336$$

$$S(x) = \begin{cases} 1 + 1.46600x + 0.25228x^3, & \text{for } x \in [0,1], \\ 2.71828 + 2.22285(x-1) + 0.75685(x-1)^2 + 1.69107(x-1)^3, & \text{for } x \in [1,2], \\ 7.38906 + 8.80977(x-2) + 5.83007(x-2)^2 - 1.94336(x-2)^3, & \text{for } x \in [2,3]. \end{cases}$$



Illustration

To approximate the integral of $f(x) = e^x$ on $[0, 3]$,

$$\int_0^3 S(x) = \int_0^1 1 + 1.46600x + 0.25228x^3 \, dx$$

$$+ \int_1^2 2.71828 + 2.22285(x-1) + 0.75685(x-1)^2 + 1.69107(x-1)^3 \, dx$$

$$+ \int_2^3 7.38906 + 8.80977(x-2) + 5.83007(x-2)^2 - 1.94336(x-2)^3 \, dx.$$

$$= 19.55229.$$

The exact value is,

$$\int_0^3 e^x \, dx = e^3 - 1 \approx 20.08553692 - 1 = 19.08553692$$